

RATIONAL GROUP ACTIONS ON AFFINE PI-ALGEBRAS

MARTIN LORENZ

ABSTRACT. Let R be an affine PI-algebra over an algebraically closed field \mathbb{k} and let G be an affine algebraic \mathbb{k} -group that acts rationally by algebra automorphisms on R . For R prime and G a torus, we show that R has only finitely many G -prime ideals if and only if the action of G on the center of R is multiplicity free. This extends a standard result on affine algebraic G -varieties. Under suitable hypotheses on R and G , we also prove a PI-version of a well-known result on spherical varieties and a version of Schelter's catenarity theorem for G -primes.

1. INTRODUCTION

1.1. This article addresses the following general question:

Suppose a group G acts by automorphisms on a ring R . When is the set $G\text{-Spec } R$ consisting of all G -prime ideals of R finite?

Recall that a proper G -stable (two-sided) ideal I of R is called *G -prime* if $AB \subseteq I$ for G -stable ideals A and B of R implies that $A \subseteq I$ or $B \subseteq I$. For a noetherian algebra R over a field \mathbb{k} and an algebraic \mathbb{k} -torus G acting rationally on R by \mathbb{k} -algebra automorphisms, the above question was stated as Problem II.10.6 in [6].

1.2. Now assume that R is an associative algebra over an algebraically closed base field \mathbb{k} and G is an affine algebraic \mathbb{k} -group that acts rationally by \mathbb{k} -algebra automorphisms on R ; see 2.2 below for a brief reminder on rational actions.

The question in 1.1 is motivated in part by the stratification of $\text{Spec } R$ that is induced by the action of G . Namely, there is a surjection

$$\text{Spec } R \twoheadrightarrow G\text{-Spec } R$$

sending P to the largest G -stable ideal of R that is contained in P , and [20, Theorem 9] gives a precise description of the fibers of this map in terms of *commutative* algebras. Hence, from a noncommutative perspective, the focus shifts to the description of $G\text{-Spec } R$, with finiteness being the optimal scenario. It turns out that, as long as the deformation parameters are chosen in a sufficiently generic manner, $G\text{-Spec } R$ is indeed finite for all quantized coordinate algebras $R = \mathcal{O}_{\mathbf{q}}(X)$ that have been analyzed in detail thus far, the acting group G typically being a suitably chosen algebraic torus. Notable examples include the (generic) quantized coordinate rings of all semisimple algebraic groups (Joseph [13], Hodges, Levasseur and Toro [11]), quantum matrices and quantum Grassmannians (Cauchon, Lenagan and others; e.g., [7], [8] and [17]). Finiteness of $G\text{-Spec } R$ has also been observed in Leavitt path algebras R , again for the action of a suitable torus G [1]. These finiteness results all depend either on finding a presentation of R as an iterated skew polynomial algebra, a class of algebras for which finiteness has been established by Goodearl and Letzter [9], [10], or else on long calculations in R . A general finiteness criterion for $G\text{-Spec } R$ is currently lacking.

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1.3. Our main focus in this note will be on the case where R is an affine PI-algebra over \mathbb{k} . This will be assumed for the remainder of the Introduction, and G will be an affine algebraic \mathbb{k} -group that acts rationally by \mathbb{k} -algebra automorphisms on R as in 1.2.

In order to give the finiteness problem in 1.1 a geometric perspective, we mention the following connection with G -orbits of rational ideals. Here, a prime ideal P of R is called *rational* if $\mathcal{C}(R/P) = \mathbb{k}$, where $\mathcal{C}(\cdot)$ denotes the center of the classical ring of quotients. Rational primes are exactly the closed points of $\text{Spec } R$; see 2.3.4 below for several equivalent characterizations of rationality. An ideal $P \in G\text{-Spec } R$ is said to be *G-rational* if $\mathcal{C}(R/P)^G = \mathbb{k}$. The subset of $\text{Spec } R$ consisting of all rational primes of R will be denoted by $\text{Rat } R$, and $G\text{-Rat } R$ will denote the subset of $G\text{-Spec } R$ consisting of all G -rational ideals. Since R satisfies the ascending chain condition for semiprime ideals (2.3.1), the Nullstellensatz (2.3.3) and the Dixmier-Mœglin equivalence (2.3.4), the following proposition is a special case of [20, Proposition 14].

Proposition 1. *Let R be an affine PI-algebra over the algebraically closed field \mathbb{k} and let G be an affine algebraic \mathbb{k} -group that acts rationally by \mathbb{k} -algebra automorphisms on R . Then the following are equivalent:*

- (i) $G\text{-Spec } R$ is finite;
- (ii) $G\text{-Rat } R$ is finite;
- (iii) G has finitely many orbits in $\text{Rat } R$;
- (iv) $G\text{-Rat } R = G\text{-Spec } R$.

Thus, the problem at hand amounts to determining when all G -primes of R are G -rational.

1.4. In studying the finiteness question 1.1 we may assume without loss that G is connected. In this case, all G -primes of R are actually prime, and hence $G\text{-Spec } R$ is the set of all G -stable prime ideals of R ; see Lemma 4 below. The main result of this note concerns the special case where G is a torus; it extends a standard result on affine algebraic G -varieties [15, II.3.3 Satz 5] to PI-algebras.

Theorem 2. *Let R be a prime affine PI-algebra over the algebraically closed field \mathbb{k} and let G be an algebraic \mathbb{k} -torus that acts rationally by \mathbb{k} -algebra automorphisms on R . Then $G\text{-Spec } R$ is finite if and only if the action of G on the center $\mathcal{Z}(R)$ is multiplicity free.*

Here, multiplicity freeness means that, for each rational character $\lambda: G \rightarrow \mathbb{k}^\times$, the weight space $\mathcal{Z}(R)_\lambda = \{r \in \mathcal{Z}(R) \mid g.r = \lambda(g)r \text{ for all } g \in G\}$ has dimension at most 1.

The proof of Theorem 2 will be given in Section 3 after deploying some auxiliary results and a generous amount of background material in Section 2. We remark that, when R is also assumed noetherian, Theorem 2 is quite a bit easier, being an immediate consequence of Proposition 7 and Lemma 8(b) below. We conclude, in Section 4, with two results for noetherian R , namely a PI-version of a standard result on spherical varieties (Proposition 10) and a version of Schelter's catenarity theorem for G -primes (Proposition 11).

Notations and conventions. All rings have a 1 which is inherited by subrings and preserved under homomorphisms. The action of the group G on the ring R will be written as $G \times R \rightarrow R$, $(g, r) \mapsto g.r$. For any ideal I of R , we will write $I:G = \bigcap_{g \in G} g.I$; this is the largest G -stable ideal of R that is contained in I . The symbol \subset denotes a proper inclusion.

2. PRELIMINARIES

2.1. Finite centralizing ring extensions. A ring extension $R \subseteq S$ is called *centralizing* if $S = RC_S(R)$ where $C_S(R) = \{s \in S \mid sr = rs \text{ for all } r \in R\}$. In this case, for any prime ideal P of S , the contraction $P \cap R$ is easily seen to be a prime ideal of R . A centralizing extension $R \subseteq S$ is called *finite*, if S is finitely generated as left or, equivalently, right R -module. By results of G. Bergman

[2], [3] (see also [26]), the classical relations of lying over and incomparability for prime ideals hold in any finite centralizing extension $R \subseteq S$:

- given $Q \in \text{Spec } R$, there exists $P \in \text{Spec } S$ such that $Q = P \cap R$ (Lying Over);
- if $P, P' \in \text{Spec } S$ are such that $P \subset P'$ then $P \cap R \subset P' \cap R$ (Incomparability).

Lemma 3. *Let $R \subseteq S$ be a finite centralizing extension of rings and let G be a group acting by automorphisms on S that stabilize R . Assume that every ideal A of S contains a finite product of primes each of which contains A . Then contraction yields a surjective map*

$$G\text{-Spec } S \twoheadrightarrow G\text{-Spec } R, \quad I \mapsto I \cap R$$

with finite fibers. In particular, if one of $G\text{-Spec } S$ or $G\text{-Spec } R$ is finite then so is the other.

Proof. First, we note that the G -primes of S are exactly the ideals of the form $P:G$ with $P \in \text{Spec } S$. Indeed, it is straightforward to check that $P:G$ is G -prime. Conversely, for any given $I \in G\text{-Spec } S$, there are finitely many $P_i \in \text{Spec } S$ (not necessarily distinct) with $I \subseteq P_i$ and $\prod_i P_i \subseteq I$. But then $I \subseteq P_i:G$ for each i and $\prod_i P_i:G \subseteq I$, whence $I = P_i:G$ for some i . In particular, each $I \in G\text{-Spec } S$ is semiprime. The group G permutes the finitely many primes of S that are minimal over I and G -primeness forces these primes to form a single G -orbit. Therefore, we may write $I = P:G$ with $P \in \text{Spec } S$ having a finite G -orbit. Similar remarks apply to the ring R , because every ideal B of R also contains a finite product of primes each of which contains B ; this follows from the fact that B contains some finite power of $BS \cap R$ by [18, Corollary 1.4].

Now let $I \in G\text{-Spec } S$ be given and let A, B be G -stable ideals of R such that $AB \subseteq I \cap R$. Then $AS = SA$ is a G -stable ideal of S and similarly for B . Since $(AS)(BS) = ABS \subseteq I$, we must have $AS \subseteq I$ or $BS \subseteq I$ and hence $A \subseteq I \cap R$ or $B \subseteq I \cap R$. Thus contraction yields a well-defined map $G\text{-Spec } S \rightarrow G\text{-Spec } R$.

For surjectivity of the contraction map, let $J \in G\text{-Spec } R$ be given and write $J = Q:G$ with $Q \in \text{Spec } R$. By Lying Over we may choose $P \in \text{Spec } S$ with $Q = P \cap R$. Putting $I = P:G$ we obtain a G -prime of S such that $J = I \cap R$.

Finally, assume $I \in G\text{-Spec } S$ contracts to a given $J \in G\text{-Spec } R$. Write $I = P:G$ with $P \in \text{Spec } S$ having a finite G -orbit. We claim that P must be minimal over the ideal JS . Indeed, suppose that $JS \subseteq P' \subset P$ for some $P' \in \text{Spec } S$. Then Incomparability gives $P \cap R \supset P' \cap R \supseteq J = \bigcap_{g \in G} g.(P \cap R)$. Since this intersection is finite and $P' \cap R$ is prime, we conclude that $g.(P \cap R) \subseteq P' \cap R$ for some $g \in G$. Hence, $g.(P \cap R) \subset P \cap R$ which is impossible. This proves minimality of P over JS . It follows that there are finitely many possibilities for P , and hence there are finitely many possibilities for I . This completes the proof of the lemma. \square

The hypothesis that every ideal of S contains a finite product of prime divisors is of course satisfied, by Noether's classical argument, if S satisfies the ascending chain condition for ideals. More importantly for our purposes, the hypothesis also holds for any affine PI-algebra S over some commutative noetherian ring by Braun's Theorem [27, 6.3.39].

2.2. Rational group actions. Let G be an affine algebraic \mathbb{k} -group, where \mathbb{k} is an algebraically closed field, and let $\mathbb{k}[G]$ denote the Hopf algebra of regular functions on G . A \mathbb{k} -vector space M is called a G -module if M is a $\mathbb{k}[G]$ -comodule; see Jantzen [12, 2.7-2.8] or Waterhouse [29, 3.1-3.2]. Writing the comodule structure map $\Delta_M: M \rightarrow M \otimes \mathbb{k}[G]$ as $\Delta_M(m) = \sum m_0 \otimes m_1$, the group G acts by \mathbb{k} -linear transformations on M via

$$g.m = \sum m_0 m_1(g) \quad (g \in G, m \in M).$$

Such G -actions, called *rational G -actions*, are in particular locally finite: the G -orbit of any $m \in M$ is contained in the finite-dimensional \mathbb{k} -subspace of M that is generated by $\{m_0\}$. If G acts rationally on M then it does so on all G -subquotients of M . Moreover, every irreducible G -submodule of M is

finite-dimensional, and the sum of all irreducible G -submodules is an essential G -submodule of M , called the *socle* of M and denoted by $\text{soc}_G M$. In the following, we will denote the set of isomorphism classes of irreducible G -modules by $\text{irr } G$ and, for each $E \in \text{irr } G$, we let

$$[R : E] \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

denote the *multiplicity* of E in R ; see [12, I.2.14].

We will be primarily concerned with the situation where G acts rationally by algebra automorphisms on a \mathbb{k} -algebra R . This is equivalent to R being a right $\mathbb{k}[G]$ -comodule algebra in the sense of [24, 4.1.2]. In the special case where $G \cong (\mathbb{k}^\times)^d$ is an algebraic torus, we have $\text{irr } G = X(G) \cong \mathbb{Z}^d$, the lattice of rational characters $\lambda: G \rightarrow \mathbb{k}^\times$. We will usually write $\Lambda = X(G)$. A rational G -action on R is equivalent to a \mathbb{Z}^d -grading $R = \bigoplus_{\lambda \in \mathbb{Z}^d} R_\lambda$ of the algebra R . This follows from the fact that $\mathbb{k}[G]$ is the group algebra $\mathbb{k}\Lambda$ of the lattice $\Lambda \cong \mathbb{Z}^d$, and $\mathbb{k}\Lambda$ -comodule algebras are the same as Λ -graded algebras; see [24, 4.1.7]. The homogeneous component of R of degree λ is the weight space

$$R_\lambda = \{r \in R \mid g.r = \lambda(g)r \text{ for all } g \in G\},$$

and $[R : \lambda] = \dim_{\mathbb{k}} R_\lambda$.

The following lemma was referred to in the Introduction.

Lemma 4. *Let R be a \mathbb{k} -algebra, where \mathbb{k} is an algebraically closed field, and let G be an affine algebraic \mathbb{k} -group that acts rationally by \mathbb{k} -algebra automorphisms on R . If $G^0 \subseteq G$ denotes the connected component of the identity, then $G^0\text{-Spec } R$ consists of the ordinary prime ideals of R that are G^0 -stable. Moreover, $G\text{-Spec } R$ is finite if and only if $G^0\text{-Spec } R$ is finite.*

Proof. For the assertion that all G^0 -primes are prime, see [19, Proposition 19(a)].

The second assertion, that $G\text{-Spec } R$ is finite if and only if $G^0\text{-Spec } R$ is so, actually holds for any (normal) subgroup $N \trianglelefteq G$ having finite index in G in place of G^0 . Putting $\mathcal{G} = G/N$, we first note that the G -primes of R are exactly the ideals of the form $P = \bigcap_{x \in \mathcal{G}} x.Q$ with $Q \in N\text{-Spec } R$. Indeed, $\bigcap_{x \in \mathcal{G}} x.Q$ is easily seen to be G -prime. Conversely, any $P \in G\text{-Spec } R$ has the form $P = P':G$ with $P' \in \text{Spec } R$ by [19, Proposition 8], and hence we may take $Q = P':G^0$. Moreover, the intersection $\bigcap_{x \in \mathcal{G}} x.Q$ determines the N -prime Q to within \mathcal{G} -conjugacy, because all $x.Q$ are N -prime ideals of R and \mathcal{G} is finite. Therefore, finiteness of $N\text{-Spec } R$ is equivalent to finiteness of $G\text{-Spec } R$. \square

2.3. Some ring theoretic background on affine PI-algebras. Let R be an affine PI-algebra over a commutative noetherian ring \mathbb{k} . The following facts are well-known.

2.3.1. Semiprime ideals. The ring R satisfies the ascending chain condition for semiprime ideals and, for each ideal I of R , there are only finitely many primes of R that are minimal over I . If I is semiprime then R/I is a right and left Goldie ring and the extended centroid of R/I , in the sense of Martindale [22], is given by $\mathcal{C}(R/I) = \mathcal{Z}(Q(R/I))$, the center of the classical ring of quotients of R/I . If I is prime then $\mathcal{C}(R/I)$ is identical to the field of fractions of $\mathcal{Z}(R/I)$ by Posner's Theorem. See [27, 6.1.30, 6.3.36'], [23, 13.6.9], [19, 1.4.2] for all this.

2.3.2. G -prime ideals. By Braun's Theorem [27, 6.3.39], every ideal I of R contains a finite product of primes that contain I . As in the proof of Lemma 3 it follows that, for any group G acting by ring automorphisms on R , the G -primes of R are exactly the ideals of the form $P:G$ with $P \in \text{Spec } R$, and P can be chosen to have a finite G -orbit. In particular, every $I \in G\text{-Spec } R$ is semiprime. The ring of G -invariants $\mathcal{C}(R/I)^G$ is a field for every $I \in G\text{-Spec } R$; see [19, Prop. 9].

2.3.3. Nullstellensatz. If \mathbb{k} is a Jacobson ring then so is R : every prime ideal of R is an intersection of primitive ideals. Moreover, if P is a primitive ideal of R then P is maximal; in fact, $\mathbb{k}/P \cap \mathbb{k}$ is a field and R/P is a finite-dimensional algebra over this field. See [27, 6.3.3].

2.3.4. Rational ideals and the Dixmier-Mæglin equivalence. Now assume that \mathbb{k} is an algebraically closed field. Recall that a prime ideal P of R is said to be *rational* if $\mathcal{C}(R/P) = \mathbb{k}$ or, equivalently, $\mathcal{Z}(R/P) = \mathbb{k}$. By Posner's Theorem [27, 6.1.30], this forces P to have finite \mathbb{k} -codimension in R . In fact, for any prime ideal of R , the following properties coincide (*Dixmier-Mæglin equivalence*), the implications \Rightarrow being either trivial or immediate from the Nullstellensatz:

finite codimensional \equiv maximal \equiv locally closed in $\text{Spec } R \equiv$ primitive \equiv rational .

2.4. The trace ring of a prime PI-ring. Let R be a prime PI-ring with center $C = \mathcal{Z}(R)$. By Posner's Theorem [27, 6.1.30], the central localization $Q(R) = R_{C \setminus \{0\}}$ is a central simple algebra over the field of fractions $F = Q(C) = \mathcal{C}(R)$. For each $q \in Q(R)$ we can consider the *reduced characteristic polynomial* $c_q(X) \in F[X]$. In detail, letting F^{alg} denote an algebraic closure of F , we have an isomorphism of F^{alg} -algebras

$$\varphi: Q(R) \otimes_F F^{\text{alg}} \cong M_n(F^{\text{alg}}) \quad (1)$$

for some n . This isomorphism allows us to define $c_q(X)$ as the characteristic polynomial of the matrix $\varphi(q \otimes 1) \in M_n(F^{\text{alg}})$. One can show that $c_q(X)$ has coefficients in F and is independent of the choice of the isomorphism φ ; see [25, §9a] or [5, §12.3].

The *commutative trace ring* of R , by definition, is the C -subalgebra of F that is generated by the coefficients of all polynomials $c_r(X)$ with $r \in R$; this algebra will be denoted by T . The *trace ring* of R , denoted by TR , is the C -subalgebra of $Q(R)$ that is generated by R and T . The following result is standard; see [23, 13.9.11] or [28, 3.2].

Lemma 5. *Let R be a prime PI-ring that is an affine algebra over some commutative noetherian ring \mathbb{k} . Then T is an affine commutative \mathbb{k} -algebra and TR is a finitely generated T -module. Furthermore, TR is finitely generated as R -module if and only if R is noetherian.*

Now suppose that a group G acts by ring automorphisms on R . The action of G extends uniquely to an action on the trace ring TR , and this action stabilizes T . To see this, note that the G -action on R extends uniquely to an action on the ring of fractions $Q(R)$. Each $g \in G$ stabilizes $F = \mathcal{Z}(Q(R))$, and hence g yields an automorphism of $F[X]$ via its action on the coefficients of polynomials. The reduced characteristic polynomials of $q \in Q(R)$ and of $g.q$ are related by

$$c_{g.q}(X) = g.c_q(X) . \quad (2)$$

Indeed, extending g to a field automorphism of F^{alg} , we obtain automorphisms $M_n(g) \in \text{Aut } M_n(F^{\text{alg}})$ and $\alpha_g \in \text{Aut } Q(R) \otimes_F F^{\text{alg}}$, the latter being defined by $\alpha_g(q \otimes f) = g.q \otimes g.f$. Fixing φ as in (1) we obtain an isomorphism of F^{alg} -algebras $M_n(g)^{-1} \circ \varphi \circ \alpha_g: Q(R) \otimes_F F^{\text{alg}} \cong M_n(F^{\text{alg}})$. Using this isomorphism to compute reduced characteristic polynomials, we see that $c_q(X) = g^{-1}.c_{g.q}(X)$, proving (2). Since $g.r \in R$ for $r \in R$, equation (2) shows that the commutative trace ring T is stable under the action of G on $Q(R)$, and hence so is the trace ring TR . For rational actions, we have the following result of Vonessen [28, Proposition 3.4].

Lemma 6 (Vonessen). *Let R a prime PI-algebra over an algebraically closed field \mathbb{k} and let G be an affine algebraic \mathbb{k} -group that acts rationally by \mathbb{k} -algebra automorphisms on R . Then the induced G -actions on TR and on T are rational as well.*

In general, the finiteness problem 1.1 transfers nicely to trace rings.

Proposition 7. *Let R be a prime PI-ring that is an affine algebra over some commutative noetherian ring. Let G be a group acting by ring automorphism on R and consider the induced G -actions on T and on TR . Then $G\text{-Spec } T$ is finite if and only if $G\text{-Spec } TR$ is finite. If R is noetherian, then this is also equivalent to $G\text{-Spec } R$ being finite.*

Proof. Lemma 3, applied to the finite centralizing extension $T \subseteq TR$ (Lemma 5), tells us that finiteness of $G\text{-Spec } TR$ is equivalent to finiteness of $G\text{-Spec } T$. If R is noetherian then we may argue in the same way for the finite centralizing extension $R \subseteq TR$. \square

3. MAIN RESULT

Throughout this section, R denotes an affine PI-algebra over an algebraically closed field \mathbb{k} and G will be an affine algebraic \mathbb{k} -group that acts rationally by \mathbb{k} -algebra automorphisms on R .

3.1. Sufficient criteria for G -rationality. By Proposition 1 we know that $G\text{-Spec } R$ is finite if and only if all G -primes of R are G -rational. Therefore, G -rationality criteria are essential. As usual, the algebra R will be called G -prime if the zero ideal of R is G -prime; similarly for G -rationality.

Lemma 8. *Assume that R is G -prime.*

- (a) *If there is an $N \in \mathbb{Z}$ such that $[\text{soc}_G \mathcal{Z}(R) : E] \leq N$ for all $E \in \text{irr } G$ then R is G -rational.*
- (b) *If G is connected solvable then R is G -rational if and only if $[\text{soc}_G \mathcal{Z}(R) : E] \leq 1$ for all $E \in \text{irr } G$.*

Proof. (a) For a given $q \in \mathcal{C}(R)^G$ put $I = \{r \in R \mid qr \in R\}$; this is a nonzero G -stable ideal of R . Therefore, $J = I^N \cap \mathcal{Z}(R)$ is a nonzero G -stable ideal of $\mathcal{Z}(R)$; see [27, 6.1.28]. Note that $q^i J \subseteq \mathcal{Z}(R)$ for $0 \leq i \leq N$. We have $E \hookrightarrow J$ for some $E \in \text{irr } G$ and multiplication with q^i yields a G -equivariant map $E \hookrightarrow J \rightarrow \mathcal{Z}(R)$. Since $\dim_{\mathbb{k}} \text{Hom}_G(E, \mathcal{Z}(R)) = [\text{soc}_G \mathcal{Z}(R) : E] \leq N$, there are $k_i \in \mathbb{k}$, not all 0, such that $c = \sum_{i=0}^N k_i q^i$ annihilates E . But nonzero elements of $\mathcal{C}(R)^G$ are regular in $Q(R)$; so we must have $c = 0$. Thus, q is algebraic over \mathbb{k} and so $q \in \mathbb{k}$.

(b) The condition is sufficient by part (a). For the converse, assume that $E_1 \oplus E_2 \subseteq \mathcal{Z}(R)$ for isomorphic $E_i \in \text{irr } G$. By the Lie-Kolchin Theorem [4, III.10.5], $E_i = \mathbb{k}x_i$ for suitable x_i . Since x_i generates a G -stable two-sided ideal, x_i is regular in R . The quotient $x_1 x_2^{-1} \in \mathcal{C}(R)$ is a non-scalar G -invariant; so R is not G -rational. \square

Remark. A simplified version of the argument in the proof of (a), without recourse to [27, 6.1.28], establishes the following general fact: Let A be an arbitrary (associative) \mathbb{k} -algebra and let G be a group that acts on A by locally finite \mathbb{k} -algebra automorphisms. If there is an $N \in \mathbb{Z}$ such that $[A : E] \leq N$ for all finite-dimensional irreducible $\mathbb{k}G$ -modules E then $G\text{-Spec } A = G\text{-Rat } A$.

3.2. Regular primes. Recall from (1) that if R is prime, then the classical ring of quotients $Q(R)$ is a central simple algebra over the field of fractions $F = Q(\mathcal{Z}(R))$. The *PI degree* of R , by definition, is the degree of this central simple algebra: $\text{PI deg } R = \sqrt{\dim_F Q(R)}$. For any $P \in \text{Spec } R$, one has $\text{PI deg } R/P \leq \text{PI deg } R$. The prime P is called *regular* if equality holds here. The regular primes form an open subset of $\text{Spec } R$. See [27, p. 104] or [23, 13.7.2] for all this.

Now let G be an algebraic \mathbb{k} -torus. In particular, G is connected and so $G\text{-Spec } R$ consists of the G -stable prime ideals of R by Lemma 4.

Lemma 9. *Let G be an algebraic \mathbb{k} -torus and assume that R is prime. Then, for every regular $P \in G\text{-Spec } R$, we have $\text{tr deg}_{\mathbb{k}} \mathcal{C}(R/P)^G \leq \text{tr deg}_{\mathbb{k}} \mathcal{C}(R)^G$. Consequently, if R is G -rational then all regular primes in $G\text{-Spec } R$ belong to $G\text{-Rat } R$.*

Proof. Let $P \in G\text{-Spec } R$ be regular. Put $n = \text{PI deg } R$ and let $g_n(R)^+$ denote the Formanek center of R ; this is a G -stable ideal of $\mathcal{Z}(R)$ such that $g_n(R)^+ \not\subseteq P$; see [27, 6.1.37] or [23, 13.7.2(i)]. Therefore, we may choose a semi-invariant $c \in g_n(R)^+$ with $c \notin P$. The group G acts rationally on localization $R_c = R[1/c]$ and R_c is Azumaya by the Artin-Procesi Theorem [23, 13.5.14]. Therefore, $\mathcal{Z}(R_c)$ maps onto $\mathcal{Z}(R_c/PR_c)$ and $\mathcal{Z}(R_c)_\lambda$ maps onto $\mathcal{Z}(R_c/PR_c)_\lambda$ for all $\lambda \in X(G)$. The map $\mathcal{Z}(R_c) \twoheadrightarrow \mathcal{Z}(R_c/PR_c)$ extends to a G -equivariant epimorphism $\mathcal{Z}(R_p) \twoheadrightarrow \mathcal{C}(R/P) =$

$Q(\mathcal{Z}(R_c/PR_c))$, where $\mathfrak{p} = P \cap \mathcal{Z}(R)$. Since $\mathcal{Z}(R_{\mathfrak{p}})^G \subseteq \mathcal{C}(R)^G$, it suffices to show that $\mathcal{Z}(R_{\mathfrak{p}})^G$ maps onto $\mathcal{C}(R/P)^G$. But, given $q \in \mathcal{C}(R/P)^G$, we can find a semi-invariant $0 \neq x \in \mathcal{Z}(R_c/PR_c)_{\lambda}$ such that $qx \in \mathcal{Z}(R_c/PR_c)$, and we can further find $y, z \in \mathcal{Z}(R_c)_{\lambda}$ with $y \mapsto x$ and $z \mapsto qx$. Then $zy^{-1} \in \mathcal{Z}(R_{\mathfrak{p}})^G$ maps to q . This proves the lemma. \square

3.3. Proof of Theorem 2. Let G be an algebraic \mathbb{k} -torus and assume that R is prime. We need to show that $G\text{-Spec } R$ is finite if and only if the action of G on $\mathcal{Z}(R)$ is multiplicity free. By Lemma 8(b), the latter property is equivalent to G -rationality of R , and this is certainly necessary for $G\text{-Spec } R$ to be finite by Proposition 1.

Now assume that R is G -rational. By Proposition 1 we must show that all G -primes of R are G -rational. Lemma 9 ensures this for the regular G -primes. In particular, we may assume that $n := \text{PI deg } R > 1$. Now consider $P \in G\text{-Spec } R$ with $\text{PI deg } R/P < n$. Then P contains the ideal $\mathfrak{a} = g_n(R)R \subseteq R$; this is a nonzero G -stable common ideal of R and of the trace ring $R' := TR$ of R ; see [27, 6.1.37 and 6.3.28]. All primes of R that are minimal over \mathfrak{a} are G -stable. Let Q be one of these primes such that $Q \subseteq P$. It suffices to show that Q is G -rational. For, then we may replace R by R/Q , and since $\text{PI deg } R/Q < n$, we may argue by induction that P is G -rational.

First, we claim that there exists $Q' \in G\text{-Spec } R'$ with $Q' \cap R = Q$. Indeed, choosing Q' to be a G -stable ideal of R' that is maximal subject to the requirement that $Q' \cap R \subseteq Q$, it is straightforward to see that Q' is G -prime. If $Q' \cap R \neq Q$ then $Q' \not\supseteq \mathfrak{a}$ by minimality of Q over \mathfrak{a} . Thus, $Q' + \mathfrak{a}$ is a G -stable ideal of R' which properly contains Q' and yet also satisfies $(Q' + \mathfrak{a}) \cap R = (Q' \cap R) + \mathfrak{a} \subseteq Q$. Since this contradicts our maximal choice of Q' , we must have $Q' \cap R = Q$ as claimed.

Next, we show that Q' is G -rational. To see this, recall from Lemma 6 that G acts rationally on the trace rings T and TR . Moreover, T is an affine commutative \mathbb{k} -algebra that is G -rational, because $Q(T)^G = \mathcal{C}(R)^G = \mathbb{k}$. Therefore, by the case $n = 1$, we know that $G\text{-Spec } T$ is finite. By Proposition 7, $G\text{-Spec } TR$ is finite as well, and in view of Proposition 1, this forces Q' to be G -rational.

Finally, we show that Q is G -rational; this will finish the proof. But $\mathcal{C}(R/Q) \subseteq \mathcal{C}(R'/Q')$ and $\mathcal{C}(R'/Q')^G = \mathbb{k}$ by the foregoing. Therefore, $\mathcal{C}(R/Q)^G = \mathbb{k}$ as desired.

4. RELATED RESULTS

In this section, R and G are as in the previous section and R is also assumed noetherian.

4.1. Actions of reductive groups. Recall from Lemma 6 that the induced G -action on the commutative trace ring T is rational. This enables us to quote results from algebraic geometry.

Proposition 10. *Assume that R is prime and that G is connected reductive. Let $F = Q(\mathcal{Z}(R))$ denote the field of fractions of the center of R , and let $F^B \subseteq F$ denote the invariant subfield of a Borel subgroup $B \leq G$. If $F^B = \mathbb{k}$ then $B\text{-Spec } R$ is finite (and hence $G\text{-Spec } R$ is finite as well).*

Proof. By Proposition 7, $B\text{-Spec } R$ is finite if and only if $B\text{-Spec } T$ is finite. Now, T is an affine commutative domain over \mathbb{k} and the field of fractions of T is F . By a standard result on spherical varieties [14, Corollary 2.6], the condition $F^B = \mathbb{k}$ implies that there are only finitely many B -orbits in $\text{Rat } T$. The latter fact is equivalent to finiteness of $B\text{-Spec } T$ by Proposition 1. This proves the proposition. \square

4.2. Catenarity. A partially ordered set (P, \leq) is said to be *catenary* if, given any two $x < x'$ in P , all saturated chains $x = x_0 < x_1 < \cdots < x_r = x'$ have the same finite length $r = r(x, x')$. The following observation, for commutative algebras, goes back to conversations that I had with R. Rentschler a long time ago; cf. [21, §3]. As usual, we let GK dim denote Gelfand-Kirillov dimension.

Proposition 11. *If the connected component of the identity of G is solvable then the poset $(G\text{-Spec } R, \subseteq)$ is catenary. In fact, every saturated chain $Q = Q_0 \subset Q_1 \subset \cdots \subset Q_r = Q'$ in $G\text{-Spec } R$ has length $r = \text{GK dim } R/Q - \text{GK dim } R/Q'$.*

Proof. First assume that G is connected; so $G\text{-Spec } R$ consists of the G -stable primes of R . In view of Schelter's catenarity theorem for $\text{Spec } R$ [27, 6.3.43], it suffices to show that any two neighbors $Q \subset P$ in $G\text{-Spec } R$ are also neighbors when viewed in $\text{Spec } R$. Passing to R/Q we may assume that the algebra R is prime and P is a minimal nonzero member of $G\text{-Spec } R$, and we need to show that P has height 1 in $\text{Spec } R$. But $P \cap \mathcal{Z}(R)$ is a nonzero G -stable ideal of $\mathcal{Z}(R)$ and hence the Lie-Kolchin Theorem provides us with a G -eigenvector $0 \neq z \in P \cap \mathcal{Z}(R)$. The ideal P is a minimal prime over (z) . For, if $(z) \subseteq P' \subset P$ for some $P' \in \text{Spec } R$ then $(z) \subseteq P':G \subset P$ and $P':G \in G\text{-Spec } R$, contradicting the fact that P is a minimal nonzero member of $G\text{-Spec } R$. Thus P is minimal over (z) as claimed, and the principal ideal theorem [23, 4.1.11] gives that P has height 1 as desired.

In general, let G^0 denote the connected component of the identity of G and put $\mathcal{G} = G/G^0$. Given G -primes $Q \subset Q'$ and a saturated chain $Q = Q_0 \subset Q_1 \subset \cdots \subset Q_r = Q'$ in $G\text{-Spec } R$, we will show that $r = \text{GK dim } R/Q - \text{GK dim } R/Q'$. To this end, write $Q_i = \bigcap_{x \in \mathcal{G}} x.P_i$ for suitable $P_i \in G^0\text{-Spec } R$ as in the proof of Lemma 4. Since these intersections are finite intersections of G^0 -primes, we can arrange that $P_0 \subset P_1 \subset \cdots \subset P_r$. This is a saturated chain in $G^0\text{-Spec } R$. For, $P_i \subset P \subset P_{i+1}$ implies $Q_i = \bigcap_{x \in \mathcal{G}} x.P_i \subset \bigcap_{x \in \mathcal{G}} x.P \subset \bigcap_{x \in \mathcal{G}} x.P_{i+1} = Q_{i+1}$ since \mathcal{G} is finite, which contradicts the fact that Q_i and Q_{i+1} are neighbors in $G\text{-Spec } R$. By the first paragraph of the proof, the chain $P_0 \subset P_1 \subset \cdots \subset P_r$ is also saturated in $\text{Spec } R$, and hence it has length equal to $r = \text{GK dim } R/P_0 - \text{GK dim } R/P_r$ by Schelter's theorem. Since $\text{GK dim } R/Q_i = \text{GK dim } R/P_i$ by [16, Corollary 3.3], the proof is complete. \square

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DEPARTMENT OF MATHEMATICS, TEMPLE UNIVERSITY, PHILADELPHIA, PA 19122

E-mail address: lorenz@temple.edu

URL: <http://www.math.temple.edu/~lorenz>